Tutorial notes - 12 (Last tutorial) whizkiddeng@gmail.com.

Definite integral.

Q1. $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \sin\frac{k}{n}\pi$

Pf: consider $x_k=\frac{k}{n}$, $k=1,2\cdots n$. n discrete points.

and $x_k-x_{k-1}=\frac{1}{n}$ for $k=2\cdots n$. so if we choose right-rectangle

we should have $x_0=0$. then the limit would be:

$$\lim_{n\to+\infty} \sum_{k=1}^{n}(x_k-x_{k-1})f(x_k) = \int_0^1 f(x)\,dx \quad \text{where } f(x)=\sin(\pi x)$$

$$= \int_0^1 \sin\pi x\,dx = -\frac{\cos\pi x}{\pi}\Big|_0^1 = \frac{2}{\pi}.$$

Q2. $\int_0^1 \frac{1}{1+x}dx$. rewrite it in limit-form.

Pf: we choose $x_0=0$, $x_k=\frac{k}{n}$, $k=1,2\cdots n$

so $x_k-x_{k-1}=\frac{1}{n}$. then:

$$\int_0^1 \frac{1}{1+x}dx = \lim_{n\to+\infty}\sum_{k=1}^{n}(x_k-x_{k-1})f(x_k) = \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{1}{1+\frac{k}{n}} = \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n+k}.$$

And $\int_0^1\frac{1}{1+x}dx = \ln(1+x)\Big|_0^1 = \ln 2$.

Q3. compute the area of an ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Pf: Due to the symmetric of ellipse, we just have to compute the area above the x-axis.

$$S = 2\int_{-a}^{a} f(x)\,dx \qquad f(x)=y=b\sqrt{1-\frac{x^2}{a^2}}$$

$$= 2b\int_{-a}^{a}\sqrt{1-\frac{x^2}{a^2}}\,dx \qquad x=a\sin t$$

$$= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt$$

$$= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos 2t+1}{2}\,dt$$

$$= ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos 2t+1)\,dt = ab\left(\frac{1}{2}\sin 2t+t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$$

$$=\pi ab.$$

Q4. Prove $\lim_{n\to\infty} \int_0^1 (1-x^2)^n dx = 0$

①. $x = \sin t$

$$\int_0^1 (1-x^2)^n dx = \underbrace{\int_0^{\frac{\pi}{2}} \cos^{2n+1} t \, dt}_{= I_{2n+1}}$$, first we choose a number $\varepsilon \in (0,1)$

this ε is arbitary, but once choosed, it is fixed, we can change it later.

then: $$\int_0^{\frac{\pi}{2}} \cos^{2n+1} t\, dt = \underbrace{\int_0^{\varepsilon} \cos^{2n+1} t\, dt}_{(1)} + \underbrace{\int_{\varepsilon}^{\frac{\pi}{2}} \cos^{2n+1} t\, dt}_{(2)}$$

for (1), $0 \le \cos t \le 1$ when $t \in (0,\varepsilon)$, so $(1) \le \int_0^{\varepsilon} 1\, dt = \varepsilon$

for (2), $\cos t$ is decreasing function in $[\varepsilon, \frac{\pi}{2}]$, so $\cos t \le \cos\varepsilon < 1$, $t \in [\varepsilon, \frac{\pi}{2}]$

so $$(2) \le \int_{\varepsilon}^{\frac{\pi}{2}} \underbrace{\cos^{2n+1}\varepsilon}_{\text{constant}}\, dt = \cos^{2n+1}\varepsilon\,(\tfrac{\pi}{2}-\varepsilon) = (\tfrac{\pi}{2}-\varepsilon)\, q^{2n+1}, \quad 0<q=\cos\varepsilon<1$$

so $I_{2n+1} = (1) + (2) \le \varepsilon + (\frac{\pi}{2}-\varepsilon) q^{2n+1}$ (3)

for $q<1$, so $\lim_{n\to+\infty} q^{2n+1} = 0$, in (3), let $n \to +\infty$:

$0 \le I \le \varepsilon$, ($I = \lim_{n\to+\infty} I_{2n+1}$) (we have to prove this limit exists) *

now we change ε, let $\varepsilon \to 0$. so $0 \le I \le 0 \Rightarrow I = 0$

so $\lim_{n\to\infty} I_{2n+1} = \lim_{n\to\infty} \int_0^1 (1-x^2)^n dx = 0$

(the above argument is not very rigorous for you may not learn the rigorous definition of the limit, if you know that definition, actually * is not necessary).

②. we try to deduce a formula for $\int_0^{\frac{\pi}{2}} \cos^n t\, dt$ and $\int_0^{\frac{\pi}{2}} \sin^n t\, dt$.

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n t\, dt = -\int_0^{\frac{\pi}{2}} \sin^{n-1} t\, d\cos t \quad \text{(integrate by parts)}$$

$$= -\sin^{n-1} t \cdot \cos t \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos t\, d\sin^{n-1} t$$

$$= 0 + \int_0^{\frac{\pi}{2}} \cos t\,(n-1)\sin^{n-2} t \cdot \cos t\, dt$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} t\, \underbrace{(1-\sin^2 t)}_{= \cos^2 t}\, dt$$

$$= (n-1) I_{n-2} - (n-1) I_n$$

So $I_n = \frac{n-1}{n} I_{n-2}$, reduction formula

when n is odd. $I_n = \frac{n-1}{n} I_{n-2} = \frac{(n-1)(n-3)}{n(n-2)} I_{n-4} = \cdots = \frac{(n-1)\cdots 2}{n \cdots 3} I_1$ $\quad (I_1 = \int_0^{\frac{\pi}{2}} \sin t \, dt = 1)$

like $I_7 = \frac{6}{7} I_5 = \frac{6\times4}{7\times5} I_3 = \frac{6\times4\times2}{7\times5\times3} I_1$

when n is even $I_n = \frac{n-1}{n} I_{n-2} = \frac{(n-1)\cdots 1}{n \cdots 2} I_0$ $\quad (I_0 = \int_0^{\frac{\pi}{2}} \sin^0 t \, dt = \frac{\pi}{2})$

Q5. $\int_{\frac{1}{2}}^{2} (1+x-\frac{1}{x}) e^{x+\frac{1}{x}} dx$

$$= \underbrace{\int_{\frac{1}{2}}^{2} e^{x+\frac{1}{x}} dx}_{= A} + \underbrace{\int_{\frac{1}{2}}^{2} (x-\frac{1}{x}) e^{x+\frac{1}{x}}}_{= B}$$

$B = \int_{\frac{1}{2}}^{2} x(1-\frac{1}{x^2}) e^{x+\frac{1}{x}} dx$ $\quad (x+\frac{1}{x})' = 1-\frac{1}{x^2}$

$= \int_{\frac{1}{2}}^{2} x \, d e^{x+\frac{1}{x}}$

$= x e^{x+\frac{1}{x}} \Big|_{\frac{1}{2}}^{2} - \int_{\frac{1}{2}}^{2} e^{x+\frac{1}{x}} dx$

$= \frac{3}{2} e^{\frac{5}{2}} - A$

So $A + B = \frac{3}{2} e^{\frac{5}{2}}$.

Differential under the integrate:

consider $\int_{A(t)}^{B(t)} f(x) dx$, this is a function depends on t. by using fundamental thm:

$G(t) = \int_{A(t)}^{B(t)} f(x) \, dx = F(B(t)) - F(A(t))$ where $F'(x) = f(x)$ is the primitive function

so $G'(t) = (F(B(t)) - F(A(t)))'$ $\quad$ (chain-rule. $F(B(t)) = F \circ B(t)$)

$= F'(B(t)) \cdot B'(t) - F'(A(t)) A'(t)$

$= f(B(t)) (B'(t)) - f(A(t)) A'(t)$ $\quad (F'(x) = f(x))$.

Q6. $\int_{-2x}^{x^3} x\sqrt{t^4+t+1}\, dt$.

Pf: $G(x) = x \int_{-2x}^{x^3} \sqrt{t^4+t+1}\, dt = x \cdot A(x)$

$G'(x) = A(x) + x\, A'(x)$

$A'(x) = \sqrt{x^{12}+x^3+1} \cdot 3x^2 - \sqrt{16x^4-2x+1} \cdot (-2)$.

Q7. $\int_{-x}^{x} |\cos t|^{\frac{7}{2}}\, dt$ $\quad$ ($0<x<\frac{\pi}{2}$)

$G(x) = \int_{-x}^{x} |\cos t|^{\frac{7}{2}} dt = \int_{-x}^{x} (\cos t)^{\frac{7}{2}} dt$ (for $\cos t > 0$ when $t \in (-x, x)$, how about $\sin t$?)

$\Rightarrow G'(x) = (\cos x)^{\frac{7}{2}} \cdot 1 - (\cos(-x)^{\frac{7}{2}})(-1)$

$= 2\cos^{\frac{7}{2}} x$.

Q8. Further exercise 17.

Pf: $\int_0^x f(t)\, dt = \int_x^1 f(t)\, dt$. for all $x \in [0,1]$

So $\int_0^x f(t)\, dt + \int_x^1 f(t)\, dt = \underline{\underline{\int_0^1 f(t) dt}}$ is a constant.

but $\int_0^x f(t) dt + \int_x^1 f(t)\, dt = 2\int_0^x f(t) dt$ a function depends on x.

$= g(x)$

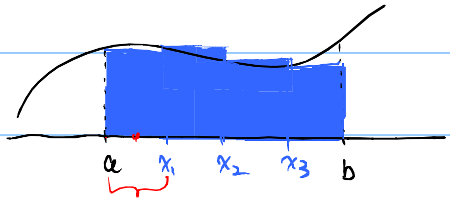
$\Rightarrow g(x) = \int_0^1 f(t)\, dt = c. \Rightarrow g'(x) = 0 = 2f(x) \Rightarrow f(x) = 0$

Tutorial 12.   Min, Jie   jmin@math.cuhk.edu.hk

* Riemann Integral

Definition: limit of Riemann sum as the partition gets finer and finer

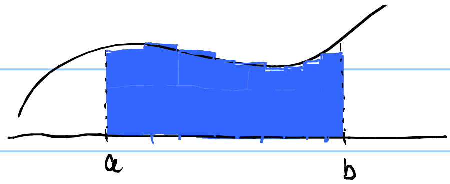
$\int_a^b f(t)\,dt$



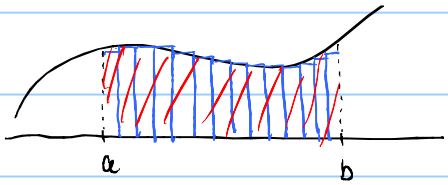
Riemann Sum

blue part $\rightsquigarrow \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k$

$\xi_k \in [x_k, x_{k+1}]$, $\Delta x_k = x_{k+1} - x_k$



as $n\to\infty$, the partition of $[a,b]$ gets finer and finer



the area of the rectangles gets closer and closer to the actual area under the graph of the function

Riemann integral
||
Area under the graph of f over $[a,b]$

Thm: $f:[a,b]\to\mathbb{R}$ cont's function $\Rightarrow$ f is Riemann integrable on $[a,b]$

$$\int_a^b f(x) = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k$$

where $\xi_k \in [x_k, x_{k+1}]$, $\Delta x_k = x_{k+1} - x_k$

$\max_{0\le k\le n-1} \Delta x_k \to 0$ as $n\to\infty$

Eg. Compute $\int_0^1 x^2\,dx$ using Riemann sum

$$= \lim_{n\to\infty}\sum_{k=0}^{n-1} f(\xi_k)\cdot\Delta x_k$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^2\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=0}^{n-1} k^2$$

$$= \lim_{n\to\infty}\frac{1}{n^3}\frac{(n-1)n(2n-1)}{6} = \frac{1}{3}$$

$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

$0 \quad \frac{1}{n} \quad \frac{2}{n} \quad \frac{3}{n} \quad -- \quad \frac{n-1}{n} \quad 1$

$\Delta x_k = \frac{1}{n}$, $\xi_k \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$

Eg. Compute $\int_a^b x^{-1}\,dx$

subdividing $[a,b]$ into $[a,aq],[aq,aq^2],\cdots[aq^{n-1},aq^n]$

where $q=\left(\frac{b}{a}\right)^{\frac{1}{n}}$

$[aq^k, aq^{k+1}],\ k=0,\cdots n-1$

$\Delta x_k = aq^k(q-1)$

$\xi_k = aq^k$

$$\int_a^b x^{-1}dx = \lim_{n\to\infty}\sum_{k=0}^{n-1}\cancel{(aq^k)}^{-1}\cancel{(aq^k)}(q-1)$$

$$= \lim_{n\to\infty} n\left(\left(\frac{b}{a}\right)^{\frac{1}{n}}-1\right)$$

$$= \lim_{n\to\infty}\frac{\left(\frac{b}{a}\right)^{\frac{1}{n}}-1}{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{b}{a}\right)^{\frac{1}{n}}\ln\left(\frac{b}{a}\right) = \ln\left(\frac{b}{a}\right)$$

L'Hopital's Rule

$\int_a^b x^{-1}dx = \left[\ln x\right]_a^b$

$= \ln b - \ln a$

$\star$ Fundamental Thm of Calculus

$f:[a,b]\to\mathbb{R}$ cont's. Let $F(x)=\int_a^x f(t)\,dt$

Then $F(x)$ is cont's on $[a,b]$, diff on (a,b)

and $\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \forall x\in(a,b)$

Eg. $\boxed{\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt}$, let $F(x)=\int_0^x f(t)\,dt$

so $\int_{g(x)}^{h(x)} f(t)dt = \int_0^{h(x)} f(t)dt - \int_0^{g(x)} f(t)dt$

Let $G(x) = F(h(x))$, $H(x) = F(g(x))$

then $\int_{g(x)}^{h(x)} f(t)\,dt = G(x) - H(x)$

so $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = \frac{d}{dx}G(x) - \frac{d}{dx}H(x)$

$= F'(h(x))h'(x) - F'(g(x))\,g'(x)$ Chain Rule.

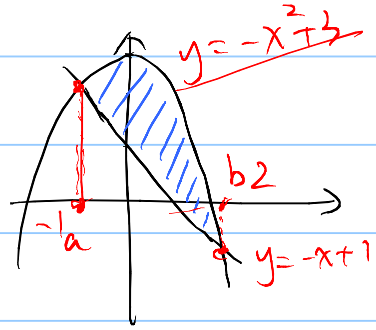
$= f(h(x))h'(x) - f(g(x))\,g'(x)$ $F'(x)=f(x)$

* Definite integral as area

Compute the regions bounded by the graph of $\int_a^b |f-g|\,dx$

$y=-x+1$ and $y=-x^2+3$

$$\begin{cases} y=-x+1 \\ y=-x^2+3 \end{cases} \Rightarrow \begin{cases} x=2, & y=-1 \\ x=-1, & y=2 \end{cases}$$



$$\text{Area} = \int_{-1}^{2} \left((-x^2+3) - (-x+1) \right) dx$$

$$= \int_{-1}^{2} (-x^2+x+2)\,dx = \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_{-1}^{2} = \frac{9}{2}$$

Exercise :

① Evaluate $\int_0^2 2^x\,dx$ using Riemann Sum.

② Find relation between $\int_0^{x^n} f(t)\,dt$ and $\int_0^x f(t)\,dt$

③ Find the area bounded by

$y^2=x$, $y^2=2-x$

 qyjiang@math.cuhk.edu.hk
Jiang Qingyuan

PLAN: General revision. exam is everything up to 5.5 of [Cheung-Lau], excluding 2.15 & 5.4

- Integration of rational functions;
- trigometric problems; etc;
- revision;

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1. Integration of rational f'ns

$\int \frac{P(x)}{Q(x)} \leadsto$ 4 types of "simplest types":

$$\frac{A}{x-a}, \quad \frac{A}{(x-a)^m}\ (m>1), \quad \frac{Ax+B}{x^2+px+q}\ (p^2-4q<0), \quad \frac{Ax+B}{(x^2+px+q)^n}\ (p^2-4q<0)\ (n>1);$$

(1) $\int \frac{A}{x-a}\,dx = A\int \frac{1}{x-a}\,d(x-a) = A\ln|x-a| + C$;

(2) $\int \frac{A}{(x-a)^m}\,dx = A\int \frac{1}{(x-a)^m}\,d(x-a) = -\frac{A}{m-1}(x-a)^{1-m} + C$;

(3) $x^2+px+q = \underbrace{\left(x+\frac{p}{2}\right)}_{=z}{}^2 + \underbrace{\left(q-\frac{p^2}{4}\right)}_{=a^2}$; then

$$\int \frac{Ax+B}{x^2+px+q}\,dx = \int \frac{Ax+B}{(x+\frac{p}{2})^2+(q-\frac{p^2}{4})}\,dx = \int \frac{A(z-\frac{p}{2})+B}{z^2+a^2}\,dz$$

$$= \int \frac{Az}{z^2+a^2}\,dz + \int \frac{B-A\cdot\frac{p}{2}}{z^2+a^2}\,dz = \frac{A}{2}\ln(z^2+a^2) + \left(B-\frac{Ap}{2}\right)\frac{1}{a}\arctan\frac{z}{a} + C$$

$$= \frac{A}{2}\ln(x^2+px+q) + \frac{2B-Ap}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}} + C;$$

(4) $z = x+\frac{p}{2}$, $x = z-\frac{p}{2}$, $a^2 := q-\frac{p^2}{4}$;

$$\int \frac{Ax+B}{(x^2+px+q)^n}\,dx = \underbrace{\int \frac{Az}{(z^2+a^2)^n}\,dz}_{\text{easy}} + \left(B-\frac{A\cdot p}{2}\right)\underbrace{\int \frac{1}{(z^2+a^2)^n}\,dz}_{\text{HARD!}}$$

easy: $\displaystyle\int \frac{\frac{A}{2}\,d(z^2+a^2)}{(z^2+a^2)^n} = \frac{A}{2}\frac{(z^2+a^2)^{1-n}}{(-n+1)} + C$

HARD! $\displaystyle I_n := \int \frac{dz}{(z^2+a^2)^n}$ need reduction:

★ reduction of. $I_n = \int \frac{dz}{(z^2+a^2)^n}$ using integration by parts: (not ~~needed~~ required for test)

$$I_n = \frac{z}{(z^2+a^2)^n} + 2n\int \frac{z^2}{(z^2+a^2)^{n+1}}\,dz = \frac{z}{(z^2+a^2)^n} + 2n\,I_n - 2na^2 I_{n+1}$$

$$\Rightarrow \quad I_{n+1} = \frac{z}{2na^2(z^2+a^2)^n} + \frac{2n-1}{2na^2} I_n;$$

especially,
$$I_2 = \int \frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\arctan\frac{x}{a} + C. \ \square$$

Example: $\displaystyle I = \int \frac{x^3+1}{x^4-3x^3+3x^2-x}\,dx$

Sol'n. $x^4-3x^3+3x^2-x = x(x-1)^3$. hence

$$\frac{x^3+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{x-1}; \qquad (★)$$

Solve A, B, C, D: [just let $x = \cdots$ & solve eq'n for A, B, C, D]

- times both sides of (★) by x, then let $x=0$; $\Rightarrow A=-1$;
- times both sides of (★) by $(x-1)^3$, let $x=1$, $\Rightarrow B=2$;
- times both sides of (★) by $(x-1)$, let $x\to-\infty$, $\Rightarrow D=2$;
- let $x=-1$, $\Rightarrow C=1$;

Hence $$\int \frac{x^3+1}{x(x-1)^3} = -\int\frac{dx}{x} + 2\int\frac{dx}{(x-1)^3} + \int\frac{dx}{(x-1)^2} + 2\int\frac{dx}{x-1} = -\frac{x}{(x-1)^2} + \ln\frac{(x-1)^2}{|x|} + C$$

$\square$

B. Trigonometric f'ns; & over all (Including substitution; & integration by parts)

Exercise B: ① $\int \frac{dx}{1-\cos x}$, ② $\int \cos^3 x\, dx$; ③ $\int \frac{dx}{\cos x \sin^2 x}$;

④ $\int \ln x\, dx$; ⑤ $\int x \arctan x\, dx$ ⑥ $\int x \cos x\, dx$;

⑦ $\int x e^{-x}\, dx$; ⑧ $\int \sin(\ln x)\, dx$;

⑨ $\int \frac{dx}{(x^2-2)(x^2+3)}$; ⑩ $\int \frac{4-2x}{(x^2+1)(x-1)^2}\, dx$;

Sol'n: ① $\int \frac{dx}{1-\cos x}$; [using $\cos 2\theta = 1-2\sin^2\theta$; & $(\cot x)' = -\frac{1}{\sin^2 x}$;]

$$= \int \frac{dx}{2\sin^2\frac{x}{2}} = -\cot\frac{x}{2} + C;$$

② $\int \cos^3 x\, dx = \int (1-\sin^2 x)\cdot d\sin x = \sin x - \frac{\sin^3 x}{3} + C$;

③ $\int \frac{dx}{\cos x \sin^2 x} dx = \int \frac{d\sin x}{(1-\sin^2 x)\sin^2 x} \overset{y=\sin x}{=\!=} \int \frac{dy}{(1-y^2)y^2} = \int \left(\frac{1}{1-y^2} + \frac{1}{y^2}\right) dy$

$$= \int \left(\frac{1}{1-y} + \frac{1}{1+y} + \frac{1}{y^2}\right) dy = \ln\left|\frac{1+y}{1-y}\right| + \frac{y^{-2+1}}{-2+1}$$

$$= \ln\left|\frac{\sin x+1}{\sin x-1}\right| - \frac{1}{\sin x} + C;$$

④ $\int \ln x\, dx \overset{\text{integration by parts}}{=\!=} x\ln x - \int x\cdot\frac{1}{x}\, dx = x\ln x - x + C$;

⑤ $\cancel{\int \frac{x}{\tan x} dx} \;\not\equiv\;$ [‼: never confuse $\tan^{-1} x := \arctan x$ with $\frac{1}{\tan x}$!]

$$\int x \arctan x\, dx = \int \arctan x\, d\frac{x^2}{2} = \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\cdot\frac{dx}{1+x^2}$$

$$= \frac{x^2}{2}\arctan x + \int \frac{1}{2}\left(\frac{1}{1+x^2} - 1\right) dx = \left(\frac{x^2}{2}+\frac{1}{2}\right)\arctan x - \frac{x}{2} + C;$$

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⑥ $\int x\cos x\,dx \overset{\text{int. by parts}}{=\!=} \int x\,d\sin x = x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + C$;

⑦ $\int xe^{-x}\,dx \overset{\text{int. by parts}}{=\!=} \begin{cases} ?\ \int e^{-x}\,dx^2 \times = x^2e^{-x} + \int x^2e^{-x}\,dx & \text{more complicated!} \\ ?\ \int -x\,de^{-x} \checkmark = -xe^{-x} - \int e^{-x}\,d(-x) \checkmark \end{cases}$

$= -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$;

has to use substitution $y = \ln x$, $dy = \frac{dx}{x}$;

⑧ $\int \sin(\ln x)\,dx == \underbrace{\int \sin(y)\,e^y\,dy}\ = I$;

two way can int. by parts:

- $I = \int \sin y\,de^y = \sin y\,e^y - \int e^y\cos y\,dy$; ☹?
- $I = \int e^y\,d(-\cos y) = -\cos y\,e^y - \int(-\cos y)\,e^y\,dy$

$= -\cos y\,e^y + \int e^y\cos y\,dy$; ☹?

☺! add them!

$\Rightarrow I = \frac{1}{2}(\sin y\,e^y - \cos y\,e^y) + C$;

$\overset{y=\ln x}{=\!=} \frac{1}{2}(\sin\ln x - \cos\ln x)\,x + C$;

□

⑨. $\int \frac{dx}{(x^2-2)(x^2+3)}$; $\frac{1}{(x^2-2)(x^2+3)} = \frac{1}{5}\left(\frac{1}{x^2-2} - \frac{1}{x^2+3}\right)$

$= \frac{1}{5}\left(\frac{1}{2\sqrt{2}}\left(\frac{1}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}\right) - \frac{1}{x^2+3}\right)$

Hence $\;// = \frac{1}{10\sqrt{2}} \ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| - \frac{1}{5\sqrt{3}} \arctan\frac{x}{\sqrt{3}} + C$;

⑩. $\int \frac{4-2x}{(x^2+1)(x-1)^2}dx$; $\frac{4-2x}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{(x-1)^2} + \frac{D}{x-1}$; (*)

Solve A, B, C, D :
- times (*) by $(x-1)^2$, let $x=1$; $\quad C = \frac{4-2}{1+1} = 1$;
- times (*) by $(x-1)$, let $x \to \infty$; $\quad A + D = 0$;
- times (*) by (x^2+1), let $x = i$; $\quad 2i+1 = Ai+B$;

$A = 2,\ B = 1$;

$\Rightarrow D = -2$;

$$\int \frac{4-2x}{(x^2+1)(x-1)^2}dx = \int \frac{2x+1}{x^2+1}dx + \int \frac{dx}{(x-1)^2} - \int \frac{2}{x-1}dx$$

$$= \ln(1+x^2) + \arctan x - (x-1)^{-1} - 2\ln|x-1| + C$$

$$= \ln \frac{x^2+1}{(x-1)^2} + \arctan x - \frac{1}{x-1} + C. \qquad \square$$

C. Revision.

C-1. Evaluate limits: (· Basic properties like sandwich thm ; L'Hôpital's rule ; & Taylor's theorem ;)

C-2. Computing Taylor series.

Prepare well for the final,
Wish everyone good luck for the exam!

Tutorial 12

Topics: Definite integral and improper integral.

Q1) Evaluate the derivative of (w.r.t. x)

a) $\int_{x}^{e^x} \ln(t)\,dt$ b) $\int_{-x^3}^{x^3} t^3 + t^2 + t + 1 \, dt$

Q2) Determine whether the improper integral is convergent.

a) $\int_{0}^{\infty} \frac{1}{x^2}\,dx$ b) $\int_{0}^{\infty} \frac{1}{x+e^x}\,dx$

Q3) Compute the following definite integral.

a) $\int_{0}^{\pi/6} \cos(x)\cos(\pi \sin(x))\,dx$ b) $\int_{1/2}^{\infty} \frac{1}{1+x^3}\,dx$

c) $\int_{0}^{\pi/2} \ln(\sin(x))\,dx$

Recall:

Let f be cont. fcn with anti-derivative F (i.e. $F' = f$).
Let $a(x)$, $b(x)$ are differentiable functions.
then $\int_{b(x)}^{a(x)} f(t)\, dt = F(a(x)) - F(b(x))$

$$\frac{d}{dx}\int_{b(x)}^{a(x)} f(t)\, dt = f(a(x))\, a'(x) - f(b(x))\, b'(x)$$

Reason: Chain rule and $F'(t) = f(t)$.

Improper integral Suppose f is a cont. fcn.

- $\int_a^{\infty} f(x)\, dx$ is convergent if $\lim_{t\to\infty}\int_a^t f(x)\, dx$ exists.
- $\int_{-\infty}^{a} f(x)\, dx$ is convergent if $\lim_{t\to-\infty}\int_t^a f(x)\, dx$ exists.

☆ $\int_{-\infty}^{\infty} f(x)\, dx$ is Convergent if both $\int_c^{\infty} f(x)\, dx$, $\int_{-\infty}^{c} f(x)\, dx$ are convergent for any $c \in \mathbb{R}$.

1a)

$$\frac{d}{dx}\int_{x}^{e^x} \ln(t)\,dt = \ln(e^x)\frac{d}{dx}e^x - \ln(x)\frac{d}{dx}x$$

$$= \ln(e^x)\, e^x - \ln(x)\cdot 1$$

$$= xe^x - \ln(x)$$

1b)

$$\frac{d}{dx}\int_{-x^3}^{x^3} t^3 + t^2 + t + 1\,dt$$

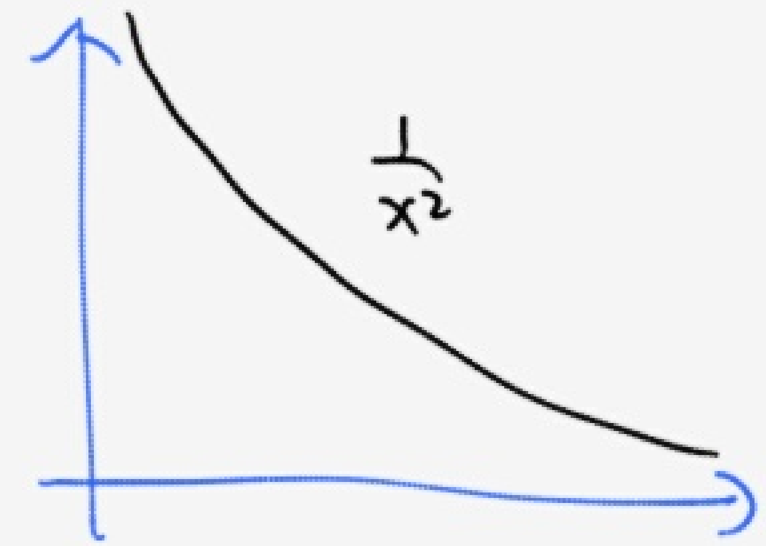
$$= \left[(x^3)^3 + (x^3)^2 + (x^3) + 1\right](3x^2) - \left[(-x^3)^3 + (-x^3)^2 + (-x^3) + 1\right](-3x^2)$$

$$= 3x^2\left[(x^9 - x^9) + (x^6 + x^6) + (x^3 - x^3) + (1+1)\right]$$

$$= 6x^8 - 6x^2$$ //

2a) Consider $\int_0^\infty \frac{1}{x^2}\,dx = \underbrace{\int_0^1 \frac{1}{x^2}\,dx}\ + \underbrace{\int_1^\infty \frac{1}{x^2}\,dx}$

are both improper



$$\int_1^t \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^t = 1 - \frac{1}{t} \longrightarrow 1 \quad \text{as } t \to \infty$$

$$\& \int_s^1 \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_s^1 = \frac{1}{s} - 1 \longrightarrow \infty \quad \text{as } s \to 0^+$$

Hence $\int_1^\infty \frac{1}{x^2}\,dx$ converges BUT $\int_0^1 \frac{1}{x^2}\,dx$ diverges.

and hence $\int_0^\infty \frac{1}{x^2}\,dx$ diverges.

2b) Consider $0 \leq \frac{1}{x+e^x} \leq \frac{1}{e^x} \quad \forall x \geq 0$

$$\Rightarrow \forall t > 0; \; 0 \leq \int_0^t \frac{1}{x+e^x}\,dx \leq \int_0^t \frac{1}{e^x}\,dx$$

$$= \int_0^t e^{-x}\,dx$$

$$= -e^{-x}\Big|_0^t$$

$$= 1 - e^{-t} \leq 1$$

Hence we have that

$\int_0^t \frac{1}{x+e^x}\,dx$ is a bounded

and increasing function of $t \geq 0$

And hence $\lim_{t\to\infty} \int_0^t \frac{1}{x+e^x}\,dx$ exists.

3a)

$$\int_0^{\pi/6} \cos(x)\cos(\pi \sin x)\,dx$$

$$= \int_0^{\pi/6} \cos(\pi \sin x)\, d\sin x \qquad \because\ d\sin x = \cos x\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi/6} \cos(\pi \sin x)\, d(\pi \sin x)$$

$$= \frac{1}{\pi}\int_0^{\pi/6} d\sin(\pi \sin x)$$

$$= \frac{1}{\pi}\left[\sin\left(\pi \sin\frac{\pi}{6}\right) - \sin(\pi \sin 0)\right]$$

$$= \frac{1}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] = \pi^{-1}$$

//

3b) Consider

$$\frac{1}{1+x^3} = \frac{1}{(1+x)(x^2-x+1)} = \frac{A}{1+x} + \frac{Bx+C}{x^2-x+1}$$

$$= \frac{A}{1+x} + \frac{B'(2x-1)}{x^2-x+1} + \frac{C'}{x^2-x+1}$$

$$\Rightarrow \quad 1 \equiv A(x^2-x+1) + B'(2x-1)(1+x) + C'(x+1)$$

by comparing the terms, we have.

$$\begin{cases} x^2: & 0 = A + 2B' \\ x: & 0 = -A + B' + C' \\ \text{Const}: & 1 = A - B' + C' \end{cases} \Rightarrow \begin{cases} A = \frac{1}{3} \\ B' = \frac{-1}{6} \\ C' = \frac{1}{2} \end{cases}$$

So $$\int_{1/2}^{\infty} \frac{1}{1+x^3}\,dx = \int_{1/2}^{\infty} \left(\frac{1}{3}\right)\left(\frac{1}{1+x}\right) + \left(\frac{-1}{6}\right)\left(\frac{2x-1}{x^2-x+1}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{x^2-x+1}\right) dx$$

$$= \frac{1}{3}\int_{1/2}^{\infty} \frac{1}{1+x}\,dx + \frac{-1}{6}\int_{1/2}^{\infty} \frac{d(x^2-x+1)}{x^2-x+1} + \frac{1}{2}\int_{1/2}^{\infty} \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \left[\frac{1}{3}\ln|1+x| + \frac{-1}{6}\ln|x^2-x+1| + \frac{1}{2}\left(\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right)\right)\right]_{1/2}^{\infty}$$

$$= \left[\frac{1}{6}\ln\left|\frac{x^2+2x+1}{x^2-x+1}\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right)\right]_{1/2}^{\infty}$$

(sub^{t}
$x-\frac{1}{2} = \frac{\sqrt{3}}{2}\tan u$)

$$= \left[0 + \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}\right)\right] - \left[\frac{1}{6}\ln 3 + 0\right] = \frac{\pi}{2\sqrt{3}} - \frac{\ln 3}{6}$$

//

3c) $\displaystyle\int_0^{\pi/2} \ln(\sin x)\,dx$

$$= \int_0^{\pi/2} \ln\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)dx = \int_0^{\pi/2} \ln 2 + \ln\sin\frac{x}{2} + \ln\cos\frac{x}{2}\,dx$$

$$= \frac{\pi}{2}\ln 2 + \int_0^{\pi/4} \ln(\sin y)(2\,dy) + \int_0^{\pi/4} \ln(\cos y)(2\,dy) \qquad \left[y = \frac{x}{2}\right]$$

$$= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln\sin y\,dy + 2\int_{\pi/2}^{\pi/4} \ln(\sin z)(-dz) \qquad \left[\begin{array}{c}\frac{\pi}{2} - z = y \\ \therefore \cos\left(\frac{\pi}{2} - z\right) = \sin z\end{array}\right]$$

$$= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln\sin x\,dx + 2\int_{\pi/4}^{\pi/2} \ln\sin x\,dx$$

$$= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/2} \ln\sin x\,dx$$

Since $\displaystyle\int_0^{\pi/2} \ln\sin x\,dx = \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/2} \ln\sin x\,dx$ Hence $\displaystyle\int_0^{\frac{\pi}{2}} \ln\sin x\,dx = -\frac{\pi}{2}\ln 2.$